Lecture 11 (1/28/22)

Return to complex analysis:

Space of analytic functions.

Given $G \subseteq \mathbb{C}$ open, $H(G)$ denotes the space of analytic (holomorphic) function in G. Clearly, $H(G) \subseteq C(G, \mathbb{C})$.

(Note: In literature, $\mathcal{O}(G)$ is often used.)

Thm 1. If $\{f_k\}_k$ is a seq. in $H(G)$ s.t. $f_k \to f$ in $C(G, \mathbb{C})$, then $f \in H(G)$. Moreover, $f_k^{(n)} \to f^{(n)}$ for all n.

Pf. $f_k \to f$ in $C(G, \mathbb{C}) \iff f_k \to f$ unif. on compacts $\Rightarrow$ f is analytic by Morera and Cauchy's Thm. DIY. Similarly, $f_k^{(n)} \to f^{(n)}$ for all n by Cauchy's estimate (or formula) as follows:

Let $K \subset\subset G$, and $\delta = (K, \mathbb{C} \setminus G) > 0$.
Let $U = \bigcup_{z \in K} B(z, \delta/2)$. Then $K \subset\subset U$,
$\overline{U} \subset\subset G$, $d(K, \mathbb{C} \setminus \overline{U}) = \delta' \geq \delta/2$
By Cauchy's Estimate in $\overline{U}$:

$$\sup_K |f^{(n)} - f^{(n)}_{k_j}| \leq \frac{n!}{(\delta')^n} \sup_{\overline{U}} |f - f_{k_j}|.$$

Since $f - f_{k_j} \to 0$ unif. on $\overline{U}$, $f^{(n)} - f^{(n)}_{k_j} \to 0$ unif. on K. $\Rightarrow f^{(n)}_{k_j} \to f^{(n)}$ in $\mathcal{C}(G, \mathbb{C})$. $\boxed{}$

<u>Cor</u>. $H(G)$ is closed in $\mathcal{C}(G, \mathbb{C})$.
($\Rightarrow (H(G), \rho)$ is complete metric space.)

<u>Thus</u>: $f_n \to f$ in $H(G) \Leftrightarrow f_n \to f$ unif. on compacts $K \subset\subset G$.

An interesting phenomenon:

region

<u>Hurwitz Thm</u>. Suppose $f_n \to f \not\equiv 0$ in $H(G)$.
Let $\overline{B(z_0,r)} \subset\subset G$ and $f \neq 0$ on $|z-z_0|=r$.
Then $\exists N$ s.t. f_n has the same # of zeros in $B(z_0,r)$ as f.

<u>Pf.</u> By Argument Princ., # zeros of f in $B(z_0,r) = m = \frac{1}{2\pi i}\int\limits_{|z-z_0|=r} \frac{f'(z)}{f(z)}\,dz$.

Since $f_n \to f$ in $H(G)$, $f_n' \to f'$ also.
In particular, $f_n \to f$, $f_n' \to f'$ unif. on $K = \{|z-z_0|=r\}$. For n large enough, $f_n \neq 0$ on K and hence $\frac{f_n'}{f_n} \to \frac{f'}{f}$ unif. on K. $\Rightarrow$

$$\frac{1}{2\pi i}\int\limits_{|z-z_0|=r} \frac{f_n'}{f_n}\,dz \to \frac{1}{2\pi i}\int\limits_{|z-z_0|=r} \frac{f'}{f}\,dz = m.$$

But $\frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f_n'}{f_n}\,dz = \#$ zeros of f_n in $B(z_0,r)$; an integer. Thus, $\exists N$ s.t. $\frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f_n'}{f_n}\,dz = m$ for $n \geq N$.

$\boxtimes$

Normal families in $H(G)$.

Def. ① A family $\mathcal{F}$ in $H(G)$ is locally bounded if $\forall z_0 \in G$ $\exists r>0$ s.t. $\overline{B(z_0,r)} \subset\subset G$ and M s.t. $|f(z)| \leq M$, $\forall z \in \overline{B(z_0,r)}$, $\forall f \in \mathcal{F}$.

Montel Thm Let $\mathcal{F} \subseteq H(G)$. Then, $\mathcal{F}$ is normal $\Leftrightarrow$ $\mathcal{F}$ is locally bdd.

Pf. Recall from AA Thm w/ $\Omega = \mathbb{C}$.

(i) $\forall z \in G$, $\overline{\{f(z) : f \in \mathcal{F}\}} \subset\subset \mathbb{C}$.

(ii) $\forall z \in G$, $\mathcal{F}$ is equicont. at z.

Montel will follow from AA by showing
(i)+(ii) $\Leftrightarrow$ locally bdd:

"$\Rightarrow$". Fix $z_0 \in G$. (i) $\Rightarrow \exists M'$ s.t. $|f(z_0)| \le M'$, $\forall f \in \mathcal{F}$. (ii) $\Rightarrow \exists r > 0$ s.t. $|f(z) - f(z_0)| < 1$ for $|z - z_0| < r$. $\Rightarrow |f(z)| \le M' + 1 = M$ for $|z - z_0| < r$. $\Rightarrow \mathcal{F}$ is locally bdd.

"$\Leftarrow$". **Locally** bdd $\Rightarrow$ (i) trivially. We claim that $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is also locally bdd. This follows easily from the <u>Cauchy</u> Estimates as in pf of Thm 1. Let $\overline{B(z_0, r)} \subset\subset G$, $M > 0$ s.t. $|f(z)| \le M$, $z \in \overline{B(z_0, r)}$, $f \in \mathcal{F}$. Then, for $z \in \overline{B(z_0, r/2)}$, $|f'(z)| \le \frac{M}{r/2}$ by C.E, i.e. $\mathcal{F}'$ locally bdd. To see that (ii) holds, note that $|f(z) - f(z_0)| \le \sup_{B(z_0, r/2)} |f'| \cdot |z - z_0|$

$$\le \frac{2M}{r} |z - z_0|$$

$\Rightarrow$ equicont. at z_0. This completes pf. $\boxtimes$